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Determination of minimum-dissipation states with self-consistent resistivity in magnetized plasmas

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A calculation of minimum-dissipation states is presented, where the resistivity profile is made consistent with the resulting current and magnetic-field profiles. Helicity balance and constant toroidal flux are imposed as constraints, and it is assumed that the resistivity and thermal conductivity have a classical dependence upon temperature, with a constant factor multiplying the latter to account for anomalous heat transport. Our results are in general agreement with the profiles expected for reversed field pinches. An iterative method is employed to calculate the resistivity and current and magnetic-field profiles that minimize the dissipation.

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A number of variational principles have been used in plasma physics to calculate the current and magnetic field profiles of relaxed, weakly turbulent plasmas. The minimum-energy principle [1] has been successful at explaining many aspects of the behavior of space and laboratory plasmas but does not allow for the inclusion of the resistivity. Thus it is impossible to include, within the framework of this model, all the effects associated with nonuniformities in the resistivity profile. In addition, it has been argued that the applicability of this principle to plasmas sustained in quasi-steady-state is questionable [2,3].

Two related principles, minimum rate of entropy production [2] and minimum rate of energy dissipation [3], have recently been proposed for dissipative plasmas sustained in quasi-steady-state. The latter has gained considerable acceptance and has been employed to calculate the current and magnetic-field profiles of plasmas sustained by the standard inductive method [3–6] and by helicity injection [7]. Although the resistivity appears explicitly in this model the situation is not completely satisfactory because its magnitude, whether uniform or variable, is taken as an external parameter independent of the current and magnetic-field profiles.

In this paper we present a model where the resistivity profile is made consistent with the current and magnetic-field profiles that minimize the dissipation rate. For simplicity we consider the case of an infinitely long plasma cylinder enclosed by a perfect conductor and sustained by an externally applied longitudinal electric field. In addition, we consider only cylindrically symmetric states ($\partial/\partial\theta\equiv 0$) and assume that the resistivity and thermal conductivity have their classical temperature dependence (with the latter multiplied by a constant factor to account for anomalous effects). This last assumption could be easily modified allowing for the use of different transport models.

We assume that the zero-order flow velocity is negligible and minimize the Ohmic dissipation rate subject to the constraints of helicity balance and constant toroidal flux. This is the same situation, except for the treatment of the resistivity, considered in Sec. III of Ref. [6]. We employ cylindrical coordinates and normalize the electric and magnetic fields, current density, and plasma radius in a similar fashion as in Ref. [6]. For the resistivity, however, we take $\eta_\perp = 2\eta_\parallel$ and normalize with $\eta_\parallel(r=0)=\eta_0$. Unless otherwise indicated normalized quantities are employed from now on.

The constant toroidal flux constraint can be written as

$$2\int B_z x\,dx = 1,\quad x = r/a\ . \tag{1}$$

The helicity balance constraint results in

$$2\int \eta\mathbf{j}\cdot\mathbf{B}x\,dx = E\ , \tag{2}$$

where $\mathbf{B}$ and $\mathbf{j}$ are related through Ampère's law, E is the applied electric field, and the resistivity η, which is a function of x, will be calculated later. The Ohmic dissipation rate is

$$P_\Omega = 2\int \frac{\eta}{B^2}[(\mathbf{j}\cdot\mathbf{B})^2 + 2|\mathbf{j}\times\mathbf{B}|^2]x\,dx\ . \tag{3}$$

Using Eqs. (1)–(3), we introduce the following functional:

$$W = 2\int \frac{\eta}{B^2}[(\mathbf{j}\cdot\mathbf{B})^2 + 2|\mathbf{j}\times\mathbf{B}|^2]x\,dx - \lambda\left[2\int \eta\mathbf{j}\cdot\mathbf{B}x\,dx - E\right] - \alpha\left[2\int B_z x\,dx - 1\right]\ ,$$

where λ and α are Lagrange multipliers associated with helicity balance and constant toroidal flux respectively.

The current and magnetic-field profiles that minimize the Ohmic dissipation rate with the constraints given above are obtained by setting the first variation of W equal to zero,

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$$\delta W = 0 \ . \tag{4}$$

Of course, it is necessary to check that the solution of Eq. (4) actually corresponds to a minimum; this is done by comparing the calculated dissipation with that corresponding to neighboring solutions. At this point the standard procedure would consist in solving the Euler-Lagrange equations obtained from (4) with a prescribed resistivity profile. Since we want to use a resistivity which is consistent with the resulting current and magnetic-field profiles an additional equation relating η to $\mathbf{j}$ and $\mathbf{B}$ is needed.

The resistivity profile is determined from momentum and energy considerations involving the plasma pressure, density, and temperature. We assume that the plasma satisfies the following steady-state heat-diffusion equation [8]:

$$\nabla\cdot(\chi\nabla T)+Q=0 \ , \tag{5}$$

where χ is the thermal conductivity, T the temperature, and Q the local heating rate. Note that Q is the coefficient of x in the integrand of Eq. (3). For χ we employ the classical ion thermal conductivity, which is much larger than the corresponding electron conductivity, multiplied by a factor f which accounts for anomalous effects. Typical values of f are between 30 and 100 for a reversed field pinch (RFP) [8]. Using SI units (temperature in eV) we write χ as

$$\chi=\frac{C_\chi n^2}{T^{1/2}B^2}, \quad C_\chi=1.6\times10^{-40}\mu^{1/2}\frac{f}{Z^2}\ln\Lambda \ . \tag{6}$$

Since there are three unknowns (n, T, and $\mathbf{B}$) and only two equations [Eqs. (4) and (5)] we need an additional equation to close the system. We assume that there is force balance and hence $\nabla p=\mathbf{j}\times\mathbf{B}$. This allows us to calculate the pressure profile by integrating $(\mathbf{j}\times\mathbf{B})_x$. Having the pressure we can write the density as

$$n=p/k_B T \ , \tag{7}$$

where k_B is the Boltzmann constant.

The resistivity is also assumed to have its classical temperature dependence

$$\eta_\parallel=\frac{C_\eta}{T^{3/2}}, \quad C_\eta=0.515\times10^{-4}Z\ln\Lambda \ . \tag{8}$$

Normalizing the pressure with $\mu_0/\langle B_z\rangle^2$, where $\langle B_z\rangle$ is the mean toroidal field, the temperature with $T(x=0)=T_0$, and substituting Eqs. (6)–(8) in Eq. (5) we obtain the following equation for the normalized resistivity:

$$\frac{d^2\eta}{dx^2}+\frac{d\eta}{dx}\left[\frac{1}{x}+2\left[\frac{\mathbf{j}\times\mathbf{B}}{p}\right]_x-\frac{2}{B}\frac{dB}{dx}\right] -\frac{A\eta}{p^2}[(\mathbf{j}\cdot\mathbf{B})^2+2(\mathbf{j}\times\mathbf{B})^2]=0 \ , \tag{9}$$

where $A=1.5C_\eta k_B^2/C_\chi$.

Equations (4) and (9) plus the force-balance condition and the constraints given in Eqs. (1) and (2) constitute a closed set that can be solved to determine the magnetic-field and resistivity profiles. The iterative method employed is as follows.

(1) Assume that the resistivity profile is known (usually a uniform profile with $\eta=1$ is employed) and solve Eqs. (4), (1), and (2) using the Rayleigh-Ritz technique [9].

(2) With the current and magnetic-field profiles determined above calculate the pressure profile and local heating rate.

(3) Using the pressure and heating-rate profiles calculated above, solve Eq. (9) to determine a *new* resistivity profile.

(4) Solve Eqs. (4), (1), and (2) again using the resistivity profile calculated above and repeat the procedure until the incremental change of the Lagrange multipliers in two successive iterations becomes less than a prescribed value (10^{-4}–10^{-5}).

The Rayleigh-Ritz method employed to solve Eqs. (4), (1), and (2) consists in expanding the unknown function, $\mathbf{B}$ in our case, as a linear combination of known localized functions with undetermined coefficients. We employed parabolic finite elements but other choices are possible. The magnetic field is written as

$$B_\theta=\sum_{i=1}^{n}\sum_{j=1}^{3}a_{ij}\phi_j^{(i)}, \quad B_z=\sum_{i=1}^{n}\sum_{j=1}^{3}c_{ij}\phi_j^{(i)} \ , \tag{10}$$

where the a_{ij} and c_{ij} are unknown coefficients and, inside each of the n intervals, the $\phi_j^{(i)}$ are defined as

$$\phi_1^{(i)}=2(u-0.5)(u-1) \ ,$$

$$\phi_2^{(i)}=4u(1-u) \ ,$$

$$\phi_3^{(i)}=2u(u-0.5) \ ,$$

with $0\le u\le 1$ and $x=u/n+(i-1)/n$, $i=1,\ldots,n$. The $6n$ unknowns, $3n$, a_{ij} and $3n$ c_{ij}, reduce to $2n+4$ when we request continuity of the function and its first derivative at the $n-1$ internal nodes. The value of the function and its first derivative at the edge of the plasma remain free, thus guaranteeing that natural boundary conditions are automatically satisfied.

Substituting Eq. (10) into Eq. (4) and using the matching conditions discussed above and the constraints given in Eqs. (1) and (2), we obtain a set of algebraic equations to determine the $2n+4$ unknown coefficients and the two Lagrange multipliers. The derivatives $\partial W/\partial a_{ij}$ and $\partial W/\partial c_{ij}$ were calculated numerically and the algebraic system solved using Newton's method. The code was checked by recovering the analytic results of Ref. [6]. The results presented here were obtained with $n=40$.

Equation (9) is a second-order differential equation which can be solved as a two-point boundary-value problem or as an initial-value problem. Since, in general, the experimental values are better known at the center of the plasma and physical arguments indicate that $d/dx=0$ at $x=0$ we prefer to solve Eq. (9) as an initial-value problem. Introducing $\psi=d\eta/dx$ we can write Eq. (9) as two first-order differential equations:

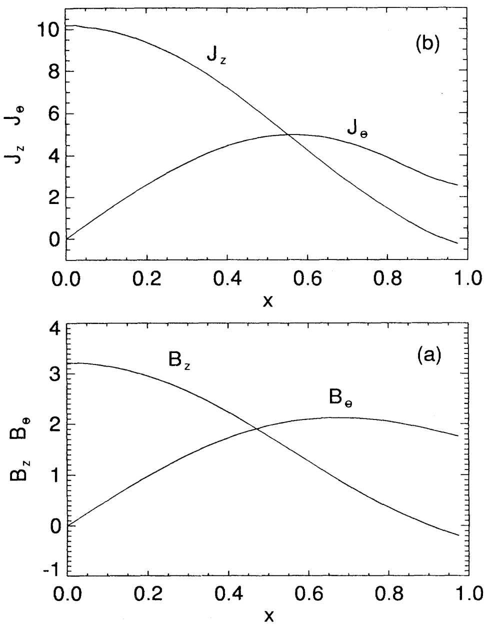

FIG. 1. (a) Both components of the magnetic field, and (b) both components of the current density for $E=18.0$, $A=1.4\times10^{-3}$, and $p(1)=0.05$.

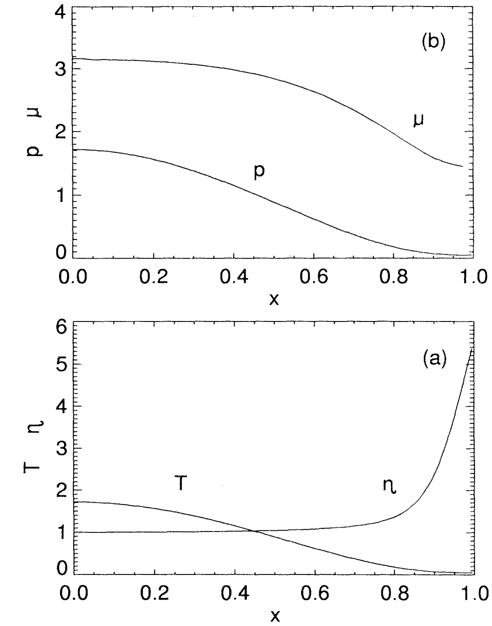

FIG. 2. (a) Resistivity and temperature profiles, and (b) pressure and $\mu(\mu\equiv\mathbf{j}\cdot\mathbf{B}/B^2)$ profiles for $E=18.0$, $A=1.4\times10^{-3}$, and $p(1)=0.05$.

$$\frac{d\eta}{dx}=\psi ,$$
$$\frac{d\psi}{dx}=-\psi\left[\frac{1}{x}+\frac{2}{p}(\mathbf{j}\times\mathbf{B})_x-\frac{2}{B}\frac{dB}{dx}\right]+\frac{A\eta}{p^2}[(\mathbf{j}\cdot\mathbf{B})^2+2(\mathbf{j}\times\mathbf{B})^2] . \tag{11}$$

This system was solved using a standard fourth-order Runge-Kutta method [10] with $\eta(0)=1$ and $\psi(0)=0$.

In Figs. 1 and 2 we present the results obtained with $E=18.0$, $A=1.4\times10^{-3}$, and $p(1)=0.05$. The value of the pressure at the edge of the plasma was taken small but finite to avoid numerical problems. Figure 1(a) shows both components of the magnetic field and Fig. 1(b) both components of the current density. Figure 2(a) shows the resistivity and temperature profiles, and Fig. 2(b) the pressure and μ ($\mu\equiv\mathbf{j}\cdot\mathbf{B}/B^2$) profiles.

The current and magnetic field profiles show the typical features of a RFP, including the reversal of B_z and j_z near the edge. In this case we have $F\equiv B_z(1)=-0.257$ and $\Theta=B_\theta(1)=1.704$. The resistivity is almost uniform in the bulk of the plasma and increases sharply near the edge. The temperature is simply $T=\eta^{-2/3}$. It is interesting to note that although the resistivity increases significantly near the edge the current density does not vanish. It is likely that including Ohm's law as a local constraint in the minimization will result in a significant reduction in the edge current [4]. The μ profile is maximum at $x=0$ and decreases towards the edge, as observed in the experiments, but does not become zero. The pressure profile has a bell-like shape with $dp/dx=0$ at both $x=0$ and 1.

The results presented above are encouraging and show that it is indeed possible to determine minimum-dissipation states employing a resistivity which is consistent with the resulting current and magnetic-field profiles. There are, however, a number of features that could be added to improve the model. These include toroidal effects, finite flow velocity, additional constraints (i.e., energy balance), and nonuniform Z_{eff}. Another interesting task would be to extend the method to the two-dimensional (2D) case and calculate the mean fields and the fluctuations corresponding to minimum-dissipation states employing a resistivity profile that depends on both the temperature and the fluctuations.

To conclude, we note that, although a classical temperature dependence was assumed for the resistivity and heat conductivity, with a multiplying factor that accounts for anomalous heat transport, other dependences can easily be considered by modifying Eq. (9). This opens the possibility of performing systematic studies where different scaling laws are employed and the results compared with the experimental observations.

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